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Darboux transformations and exact solutions of two-dimensional $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations

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Abstract

For the two-dimensional Toda equations corresponding to the Kac–Moody algebras $C_l^{(1)}$ and $D_{l+1}^{(2)}$, the Darboux transformations which keep all the reductions of the Lax pairs are constructed. The lowest degrees of the Darboux transformations are $2l + 2$ for $C_l^{(1)}$ and $2l$ for $D_{l+1}^{(2)}$. Exact solutions of these Toda equations are presented in a purely algebraic way.

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1. Introduction

The two-dimensional Toda equations [1, 2] are typical integrable systems which have been studied by many authors [3–8]. They have also important applications in physics for nonlinear lattice, as well as in differential geometry for affine spheres, Laplace sequences, harmonic tori etc [3, 9–16].

For any Kac–Moody algebra g of affine type, there is a two-dimensional Toda equation, which is written as

$$w_{k,xt} = A_k \exp\left(\sum_{i=1}^{n} c_{ki} w_i\right) - A_0 v_k \exp\left(\sum_{i=1}^{n} c_{0i} w_i\right) \quad (k = 1, \ldots, n) \tag{1}$$

where $C = (c_{ij})_{0\leqslant i,j\leqslant n}$ is the generalized Cartan matrix of g, $v = (v_0, v_1, \ldots, v_n)^T$ is a nonzero vector such that $Cv = 0$, and $A_0, \ldots, A_n$ are real constants [16, 17]. Various Kac–Moody algebras correspond to various boundary conditions of the two-dimensional Toda equations. It was shown in [18] that these equations are integrable and the Lax pairs were presented. When $g = A_l^{(1)}$, the Toda equation is periodic. Its Lax pair has a unitary symmetry and a cyclic symmetry of order l, and the Darboux transformation keeping these symmetries was presented by [5], and its geometric applications can be found in [3, 10–12].

There are also some work for other Kac–Moody algebras [19–22], although the symmetries are more complicated. In [21], the binary Darboux transformations for two-dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations were obtained from the periodic reductions of the binary Darboux transformations for A_∞, B_∞ and C_∞ Toda equations. In terms of the binary Darboux transformation, the solutions of the Toda equations are expressed by some integrals of the solutions of the Lax pairs. In order to get the explicit solutions which are purely algebraically expressed by the solutions of the Lax pair, usual Darboux transformations (without integrals) are necessary. In [23], explicit solutions of the two-dimensional $A_2^{(2)}$ Toda equation (also called the Tzitzeica equation) were presented for real spectral parameters. In [24], explicit solutions of the two-dimensional $A_{2l}^{(2)}$ Toda equation were obtained for both real and complex spectral parameters. In that case, the order of the matrices in the Lax pair is $2l + 1$ and the Lax pair has a unitary symmetry, a reality symmetry and a cyclic symmetry of order $2l + 1$. However, the number of independent functions is only l. In order to keep all these symmetries, the order of the Darboux transformation is at least $4l + 2$ when all the spectral parameters are complex.

In the present paper, we consider a $2n \times 2n$ Lax pair which corresponds to the two-dimensional Toda equations with Kac–Moody algebras $g = C_l^{(1)} (n = l + 1)$ and $D_{l+1}^{(2)}$ $(n = l)$. It has also a unitary symmetry, a reality symmetry and a cyclic symmetry of order $2n$, and the number of independent functions is only l. Apart from the order of the matrices, the Lax pairs are similar to that for the $A_{2l}^{(2)}$ case. However, in this even order case, the spectrum in the construction of Darboux transformation is different from that in the odd-order case. This leads to a different construction of the Darboux transformations.

In section 2, we discuss a general Lax pair with matrices of even order which includes exactly the two-dimensional $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations. Explicit form of the corresponding evolution equations are presented in section 3. In section 4, the Darboux transformation of degree $2n$ is constructed and the exact solutions are written down in an explicit way.

2. Structure of Lax pair

In this paper, all the matrices in the Lax pairs are of order $2n$.

For any $2n \times 2n$ matrix A or any $2n$-dimensional vector v, and for any integers i and j, define $A_{ij} = A_{i'j'}$ and $v_i = v_{i'}$ where $i \equiv i' \bmod 2n$, $j \equiv j' \bmod 2n$ and $1 \leqslant i', j' \leqslant 2n$. Especially, δ_{ij} equals 1 if $i \equiv j \bmod 2n$ and equals 0 otherwise.

Let $\omega = \mathrm{e}^{\pi \mathrm{i}/n}$, $\Omega = \mathrm{diag}(1, \omega^{-1}, \ldots, \omega^{-2n+1})$. Let m be a fixed integer. Let $K = (K_{ij}) = (\delta_{i,m-j})_{2n\times 2n}$, $J = (J_{ij}) = (\delta_{i,j-1})_{2n\times 2n}$ be constant matrices. Then K is symmetric and $\Omega^* K = \omega^{m-2} K\Omega$ where Ω^* refers to the Hermitian conjugate of Ω. Let

$$P = (p_i(x,t)\delta_{ij})_{1\leqslant i,j\leqslant 2n}, \qquad Q = (q_j(x,t)\delta_{i,j+1})_{1\leqslant i,j\leqslant 2n} \tag{2}$$

where p_i's and q_i's are real functions satisfying

$$p_{m-i} = -p_i, \qquad q_{m-1-i} = q_i \qquad (i = 1, 2, \ldots, 2n). \tag{3}$$

Then the following relations hold

$$\begin{aligned} &\bar{J} = J, && \Omega J \Omega^{-1} = \omega J, && K J K^{-1} = J^T, \\ &\bar{P} = P, && \Omega P \Omega^{-1} = P, && K P K^{-1} = -P^T, \\ &\bar{Q} = Q, && \Omega Q \Omega^{-1} = \omega^{-1} Q, && K Q K^{-1} = Q^T. \end{aligned} \tag{4}$$

Remark 1. It is easy to check that for the given constant matrix $J = (\delta_{i,j-1})_{1\leqslant i,j\leqslant 2n}$, (4) is equivalent to (2) and (3).

Denote $U(x,t,\lambda) = \lambda J + P(x,t)$, $V(x,t,\lambda) = \lambda^{-1}Q(x,t)$, then the relations (4) are equivalent to

$$\begin{aligned}
&\overline{U(x,t,\lambda)} = U(x,t,\bar\lambda), && \overline{V(x,t,\lambda)} = V(x,t,\bar\lambda)\\
&\Omega U(x,t,\lambda)\Omega^{-1} = U(x,t,\omega\lambda), && \Omega V(x,t,\lambda)\Omega^{-1} = V(x,t,\omega\lambda),\\
&KU(x,t,\lambda)K^{-1} = -(U(x,t,-\bar\lambda))^*, && KV(x,t,\lambda)K^{-1} = -(V(x,t,-\bar\lambda))^*.
\end{aligned} \tag{5}$$

These relations mean that U and V satisfy a reality symmetry, a cyclic symmetry of order $2n$ and a unitary symmetry with respect to the metric given by K.

Now we consider the Lax pair

$$\Phi_x = U(x,t,\lambda)\Phi = (\lambda J + P(x,t))\Phi, \qquad \Phi_t = V(x,t,\lambda)\Phi = \lambda^{-1}Q(x,t)\Phi \tag{6}$$

and its integrability conditions

$$Q_x = [P, Q], \qquad P_t + [J, Q] = 0. \tag{7}$$

Written in terms of p_j's and q_j's, the integrability conditions (7) become

$$q_{i,x} = (p_{i+1} - p_i)q_i, \qquad p_{i,t} = q_{i-1} - q_i \quad (i = 1,\ldots,2n). \tag{8}$$

These are the evolution equations which will be discussed in this paper. They include exactly the two-dimensional $C_l^{(1)}$ $(n = l+1)$ and $D_l^{(2)}(n = l)$ Toda equations, as will be seen in the next section.

Remark 2. Equation (8) is compatible with the constraints (3).

Remark 3. For any $2n \times 2n$ matrix $A = (A_{ij})$, let

$$\widetilde{A} = (\widetilde{A}_{ij})_{1\leqslant i,j\leqslant 2n} = (A_{i+1,j+1})_{1\leqslant i,j\leqslant 2n},$$

then $K_{ij} = \delta_{i,m-j}$ implies $\widetilde{K}_{ij} = \delta_{i,m-2-j}$. $\Omega^*K = \omega^{m-2}K\Omega$ implies $\widetilde{\Omega}^*\widetilde{K} = \omega^{m-2}\widetilde{K}\widetilde{\Omega}$ automatically. Since $\widetilde{\Omega}_{ij} = \Omega_{i+1,j+1} = \omega^{-1}\Omega_{ij}$, $\widetilde{\Omega}^*\widetilde{K} = \omega^{m-2}\widetilde{K}\widetilde{\Omega}$ implies $\Omega^*\widetilde{K} = \omega^{m-4}\widetilde{K}\Omega$. Under the transformation $(\Omega, K, J, P, Q, m) \to (\Omega, \widetilde{K}, \widetilde{J}, \widetilde{P}, \widetilde{Q}, m-2)$, the evolution equations (8) are equivalent (only the subscripts in p_j's, q_j's are changed). Therefore, we only need to consider the cases $m = 0$ and 1.

In order to construct Darboux transformations in the next section, we need the following lemma.

Lemma 1. *Suppose* $\mu \in \mathbf{C}\backslash\{0\}$.

(i) If $\Phi(x,t)$ is a solution of (6) for $\lambda = \mu$, then $\bar\Phi(x,t)$ is a solution of (6) for $\lambda = \bar\mu$.
(ii) If $\Phi(x,t)$ is a solution of (6) for $\lambda = \mu$, then for any integer k, $\Omega^k\Phi(x,t)$ is a solution of (6) for $\lambda = \omega^k\mu$.
(iii) If $\Phi(x,t)$ is a solution of (6) for $\lambda = \mu$, then $\Omega^n\bar\Phi$ is a solution of (6) for $\lambda = -\bar\mu$.
(iv) If $\Phi(x,t)$ is a solution of (6) for $\lambda = \mu$, then $\Psi = K\Omega^n\Phi$ is a solution of the adjoint Lax pair

$$\Psi_x = -U(\mu)^T\Psi, \qquad \Psi_t = -V(\mu)^T\Psi. \tag{9}$$

Therefore, $(\Phi^T K\Omega^n\Phi)_x = 0$, $(\Phi^T K\Omega^n\Phi)_t = 0$.

Proof. These result follow from (5) directly. □

3. Explicit form of the evolution equations

According to remark 3, we only need to consider the evolution equations (8) with $m = 0$ and 1.

3.1. Case $C_l^{(1)}$: $n = l + 1, m = 0$

$$\begin{aligned} &p_i = -p_{2l+2-i} = u_{i,x}, \qquad (1 \leqslant i \leqslant l), \qquad p_{l+1} = p_{2l+2} = 0,\\ &q_i = q_{2l+1-i} = \mathrm{e}^{u_{i+1}-u_i} \quad (1 \leqslant i \leqslant l-1),\\ &q_l = q_{l+1} = \mathrm{e}^{-u_l}, \qquad q_{2l+1} = q_{2l+2} = \mathrm{e}^{u_1}. \end{aligned} \tag{10}$$

When $l \geqslant 2$, the evolution equations are

$$\begin{aligned} u_{1,xt} &= \mathrm{e}^{u_1} - \mathrm{e}^{u_2-u_1},\\ u_{j,xt} &= \mathrm{e}^{u_j-u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant l-1),\\ u_{l,xt} &= \mathrm{e}^{u_l-u_{l-1}} - \mathrm{e}^{-u_l}. \end{aligned} \tag{11}$$

Let $w_j = -(u_1 + \cdots + u_j)(j = 1, \ldots, l-1)$ and $w_l = -\frac{1}{2}(u_1 + \cdots + u_l)$, then $(w_1, \ldots, w_l)$ satisfies

$$\begin{aligned} w_{1,xt} &= -\mathrm{e}^{-w_1} + \mathrm{e}^{2w_1-w_2},\\ w_{j,xt} &= -\mathrm{e}^{-w_1} + \mathrm{e}^{2w_j-w_{j-1}-w_{j+1}} \qquad (j = 2, \ldots, l-2),\\ w_{l-1,xt} &= -\mathrm{e}^{-w_1} + \mathrm{e}^{2w_{l-1}-w_{l-1}-2w_l},\\ w_{l,xt} &= -\tfrac{1}{2}\,\mathrm{e}^{-w_1} + \tfrac{1}{2}\,\mathrm{e}^{2w_l-w_{l-1}}, \end{aligned} \tag{12}$$

which is the two-dimensional Toda equation corresponding to the generalized Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & & & & \\ -2 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix} \tag{13}$$

of $C_l^{(1)}$ and $C\left(\frac{1}{2}, 1, \ldots, \frac{1}{2}\right)^T = 0$.

3.2. Case $D_{l+1}^{(2)}$: $n = l, m = 1$

$$\begin{aligned} &p_i = -p_{2l+1-i} = u_{i,x}, \quad (1 \leqslant i \leqslant l),\\ &q_i = q_{2l-i} = \mathrm{e}^{u_{i+1}-u_i} \qquad (1 \leqslant i \leqslant l-1),\\ &q_l = \mathrm{e}^{-2u_l}, \qquad q_{2l} = \mathrm{e}^{2u_1}. \end{aligned} \tag{14}$$

When $l \geqslant 2$, the evolution equations are

$$\begin{aligned} u_{1,xt} &= \mathrm{e}^{2u_1} - \mathrm{e}^{u_2-u_1},\\ u_{j,xt} &= \mathrm{e}^{u_j-u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant l-1),\\ u_{l,xt} &= \mathrm{e}^{u_l-u_{l-1}} - \mathrm{e}^{-2u_l}. \end{aligned} \tag{15}$$

Let $w_j = -(u_1 + \cdots + u_j)(j = 1, \ldots, l)$, then $(w_1, \ldots, w_l)$ satisfies

$$\begin{aligned} w_{1,xt} &= -\,\mathrm{e}^{-2w_1} + \mathrm{e}^{2w_1-w_2},\\ w_{j,xt} &= -\,\mathrm{e}^{-2w_1} + \mathrm{e}^{2w_j-w_{j-1}-w_{j+1}} \quad (j = 2, \ldots, l-1),\\ w_{l,xt} &= -\,\mathrm{e}^{-2w_1} + \mathrm{e}^{2w_l-2w_{l-1}}, \end{aligned} \tag{16}$$

which is the two-dimensional Toda equation corresponding to the generalized Cartan matrix

$$C = \begin{pmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix} \tag{17}$$

of $D_{l+1}^{(2)}$ and $C(1, 1, \ldots, 1, 1)^T = 0$.

4. Darboux transformations and exact solutions

Now we construct the Darboux transformation which keeps all the symmetries in (5).

Let μ be a nonzero complex number such that $\arg(\mu) \notin \left\{\frac{k\pi}{2n} \middle| k \in Z\right\}$, $\lambda_j = \omega^{j-1}\mu$ $(j = 1, 2, \ldots, 2n)$. Then all λ_j and $-\bar{\lambda}_j$ $(j = 1, 2, \ldots, 2n)$ are distinct. Let H be a $2n \times n$ matrix solution of (6) for $\lambda = \mu$ such that $H^T K \Omega^n H = 0$ at certain point (x_0, t_0). Then (iv) of lemma 1 implies that $H^T K \Omega^n H = 0$ holds identically. Let $H_j = \Omega^{j-1} H$ $(j = 1, 2, \ldots, 2n)$. Then (ii) of lemma 1 implies that H_j is a solution of (6) for $\lambda = \lambda_j$ $(j = 1, 2, \ldots, 2n)$.

Remark 4. Considering lemma 1, if μ is a spectral parameter in constructing Darboux transformation, it is natural to take all $\omega^{j-1}\mu$ and $\overline{\omega^{j-1}\mu}$ $(j = 1, \ldots, 2n)$ so that the reality symmetry and the cyclic symmetry are kept. However, since $\overline{\omega^{j-1}\mu} = -\overline{\omega^{j+n-1}\mu}$, according to the standard construction of unitary Darboux transformation, $\overline{\omega^{j-1}\mu}$ cannot be specified independently in the spectrum of Darboux transformation. Hence we choose only n spectral parameters $\lambda_j = \omega^{j-1}\mu$ $(j = 1, \ldots, 2n)$ here.

In this way, the Darboux transformation cannot be constructed only in terms of some column solutions of the Lax pair. Instead, the above $2n \times n$ matrix solutions H_j's are necessary in the construction.

According to (iii) of lemma 1, if H is a $2n \times n$ matrix solution of the Lax pair with spectral parameter λ, then $\Omega^n \bar{H}$ is a solution of the Lax pair with $-\bar{\lambda}$. Again, by the standard construction of unitary Darboux transformation, we want that these two matrices are orthogonal with respect to the metric defined by K, i.e. $H^T \Omega^n K H = (\Omega^n \bar{H})^* K H = 0$.

By a slight generalization of the method in [25, 26], the Darboux transformation is constructed as follows. When $\det \Gamma \neq 0$, denote

$$\Gamma_{ij} = \frac{H_i^* K H_j}{\bar{\lambda}_i + \lambda_j} \tag{18}$$

to be $n \times n$ matrices $(i, j = 1, 2, \ldots, 2n)$, $\Gamma = (\Gamma_{ij})_{1\leqslant i,j\leqslant 2n}$. Denote $\Gamma^{-1} = (\check{\Gamma}_{ij})_{1\leqslant i,j\leqslant 2n}$ where $\check{\Gamma}_{ij}$'s are $n \times n$ matrices. Let

$$G(x, t, \lambda) = \prod_{s=1}^{2n} (\lambda + \bar{\lambda}_s) \left(1 - \sum_{i,j=1}^{2n} \frac{H_i \check{\Gamma}_{ij} H_j^* K}{\lambda + \bar{\lambda}_j} \right). \tag{19}$$

Then it can be checked directly that

$$G(x, t, \lambda)^{-1} = \prod_{s=1}^{2n} (\lambda + \bar{\lambda}_s)^{-1} \left(1 + \sum_{i,j=1}^{2n} \frac{H_i \check{\Gamma}_{ij} H_j^* K}{\lambda - \lambda_i} \right). \tag{20}$$

$G(x,t,\lambda)$ is a polynomial of λ of degree $2n$ with $2n\times 2n$ matrix coefficients. Write

$$G(x,t,\lambda)=\sum_{j=0}^{2n}(-1)^j G_{2n-j}(x,t)\lambda^j, \quad G_0(x,t)=I_{2n}, \tag{21}$$

and define

$$\begin{aligned}\widetilde{U}(x,t,\lambda)&=G(x,t,\lambda)U(x,t,\lambda)G(x,t,\lambda)^{-1}+G_x(x,t,\lambda)G(x,t,\lambda)^{-1},\\ \widetilde{V}(x,t,\lambda)&=G(x,t,\lambda)V(x,t,\lambda)G(x,t,\lambda)^{-1}+G_t(x,t,\lambda)G(x,t,\lambda)^{-1}.\end{aligned} \tag{22}$$

Then for any solution Φ of (6), $\widetilde{\Phi}=G\Phi$ satisfies $\widetilde{\Phi}_x=\widetilde{U}\widetilde{\Phi}$, $\widetilde{\Phi}_t=\widetilde{V}\widetilde{\Phi}$.

Lemma 2. $\widetilde{U}=\lambda J+\widetilde{P}$, $\widetilde{V}=\lambda^{-1}\widetilde{Q}$ *where* $\widetilde{P}=P+[J,G_1]$, $\widetilde{Q}=G_{2n}QG_{2n}^{-1}$.

Proof. The lemma is obtained by direct computation. □

Lemma 3.

$$G(x,t,-\bar{\lambda})^*KG(x,t,\lambda)=\prod_{s=1}^{2n}(\bar{\lambda}_s+\lambda)(\lambda_s-\lambda)K. \tag{23}$$

Proof. The equality (23) follows from

$$G(x,t,-\bar{\lambda})^*=KG(x,t,\lambda)^{-1}K^{-1}\prod_{s=1}^{2n}(\bar{\lambda}_s+\lambda)(\lambda_s-\lambda), \tag{24}$$

which is a direct result of (19), (20) and the fact that Γ is Hermitian. □

Lemma 4.

$$\Omega G(x,t,\omega^{-1}\lambda)\Omega^{-1}=G(x,t,\lambda). \tag{25}$$

Proof.

$$\Gamma_{ij}=\frac{H^*(\Omega^*)^{i-1}K\Omega^{j-1}H}{\omega^{-i+1}\bar{\mu}+\omega^{j-1}\mu}=\omega^{(m-1)(i-1)}\frac{H^*K\Omega^{i+j-2}H}{\bar{\mu}+\omega^{i+j-2}\mu}. \tag{26}$$

From (26), $\Gamma_{i+1,j-1}=\omega^{m-1}\Gamma_{ij}$, i.e. $\hat{J}\Gamma\hat{J}=\omega^{m-1}\Gamma$ where $\hat{J}=J\otimes I_n$. It leads to $\hat{J}\Gamma^{-1}\hat{J}=\omega^{m-1}\Gamma^{-1}$, i.e. $\check{\Gamma}_{i+1,j-1}=\omega^{m-1}\check{\Gamma}_{ij}$. Substituting into (19), we get the result in the lemma. □

Lemma 5.

$$\overline{G(x,t,\bar{\lambda})}=G(x,t,\lambda). \tag{27}$$

Proof. It is not easy to verify the result directly from (19), because for each λ_j, its complex conjugate is not taken as a spectral parameter in constructing Darboux transformation. However, apart from the expression (19), there is another equivalent way of constructing Darboux matrix [27, 28], i.e. G is uniquely determined by $G(x,t,\lambda_k)H_k=0$, $G(x,t,-\bar{\lambda}_k)\Omega^n\bar{H}_k=0\,(k=1,\ldots,2n)$. Here we follow this procedure to prove the lemma.

According to (19), for $k = 1, 2, \ldots, 2n$,

$$G(x,t,\lambda_k)H_k = \prod_{s=1}^{2n}(\lambda_k + \bar{\lambda}_s)\left(H_k - \sum_{i,j=1}^{2n}\frac{H_i\check{\Gamma}_{ij}H_j^* K H_k}{\bar{\lambda}_j + \lambda_k}\right) = 0,$$

$$\begin{aligned} G(x,t,-\bar{\lambda}_k)\Omega^n\bar{H}_k &= -\prod_{s\neq k}(\bar{\lambda}_s - \bar{\lambda}_k)\sum_{i=1}^{2n}H_i\check{\Gamma}_{ik}H_k^* K\Omega^n\bar{H}_k \\ &= -\omega^{(m-2)(k-1)}\prod_{s\neq k}(\bar{\lambda}_s - \bar{\lambda}_k)\sum_{i=1}^{2n}H_i\check{\Gamma}_{ik}\overline{H^T K\Omega^n H} = 0. \end{aligned} \tag{28}$$

Now we consider $\overline{G(x,t,\bar{\lambda})}$. Since $\lambda_{j+n} = -\lambda_j$,

$$\begin{aligned} \overline{G(x,t,\bar{\lambda}_k)}H_k &= -\prod_{s\neq k+n}(\lambda_s - \lambda_{k+n})\sum_{i=1}^{2n}\bar{H}_i\overline{\check{\Gamma}}_{i,k+n}H_{k+n}^T K H_k \\ &= -\omega^{-(m-2)(k+n-1)}\prod_{s\neq k+n}(\lambda_s - \lambda_{k+n})\sum_{i=1}^{2n}\bar{H}_i\overline{\check{\Gamma}}_{i,k+n}H^T K\Omega^n H = 0. \end{aligned} \tag{29}$$

On the other hand,

$$\begin{aligned} \overline{G(x,t,-\lambda_k)}\Omega^n\bar{H}_k &= \prod_{s=1}^{2n}(\lambda_s - \bar{\lambda}_k)\overline{\left(\Omega^n H_k - \sum_{i,j=1}^{2n}\frac{H_i\check{\Gamma}_{ij}H_j^* K\Omega^n H_k}{\bar{\lambda}_j - \lambda_k}\right)} \\ &\underline{\underline{j\to j+n}}\prod_{s=1}^{2n}(\lambda_s - \bar{\lambda}_k)\overline{\left(\Omega^n H_k + \omega^{(m-2)n}\sum_{i,j=1}^{2n}\frac{H_i\check{\Gamma}_{i,j+n}H_j^* K H_k}{\bar{\lambda}_j + \lambda_k}\right)} \\ &= \prod_{s=1}^{2n}(\lambda_s - \bar{\lambda}_k)\overline{\left(\Omega^n H_k + (-1)^m\sum_{i,j=1}^{2n}H_i\check{\Gamma}_{i,j+n}\Gamma_{j,k}\right)} = 0 \end{aligned} \tag{30}$$

since $\Gamma_{i+n,j+n} = (-1)^{m-1}\Gamma_{ij}$.

Let $\Delta(x,t,\lambda) = G(x,t,\lambda) - \overline{G(x,t,\bar{\lambda})}$ and write

$$\Delta(x,t,\lambda) = \sum_{j=0}^{2n-1}\Delta_{2n-j}(x,t)\lambda^j, \tag{31}$$

then $\Delta(x,t,\lambda_k)H_k = 0$, $\Delta(x,t,-\bar{\lambda}_k)\Omega^n\bar{H}_k = 0\,(k = 1,\ldots,2n)$. Written in components, they are

$$\sum_{j=0}^{2n-1}\Delta_{2n-j}H_k\lambda_k^j = 0, \qquad \sum_{j=0}^{2n-1}\Delta_{2n-j}\Omega^n\bar{H}_k(-\bar{\lambda}_k)^j = 0, \tag{32}$$

$(k = 1,\ldots,2n)$, i.e.

$$(\Delta_{2n}, \Delta_{2n-1}, \ldots, \Delta_1)\mathcal{M} = 0 \tag{33}$$

where $\mathcal{M} = (\mathcal{M}_{ij})_{1\leqslant i\leqslant 2n, 1\leqslant j\leqslant 4n}$ with $\mathcal{M}_{ij} = \lambda_j^{i-1}H_j$ and $\mathcal{M}_{i,2n+j} = (-\bar{\lambda}_j)^{i-1}\Omega^n\bar{H}_j$ for $i, j = 1,\ldots,2n$.

Define the block matrix $\mathcal{N} = (\mathcal{N}_{ij})_{1\leqslant i\leqslant 4n, 1\leqslant j\leqslant 2n}$ with

$$\mathcal{N}_{ij} = (-\bar{\lambda}_i)^{-j+1}H_i^* K \tag{34}$$

and $\mathcal{N}_{i,2n+j} = \lambda_i^{-j+1} H_i^T K \Omega^n$ for $i, j = 1, \dots, 2n$. Then

$$\begin{aligned}\sum_{k=1}^{2n} \mathcal{N}_{ik}\mathcal{M}_{kj} &= \sum_{k=1}^{2n} (-\bar\lambda_i)^{-k+1}\lambda_j^{k-1} H_i^* K H_j \\ &= \frac{1-(-\lambda_j/\bar\lambda_i)^{2n}}{1+\lambda_j/\bar\lambda_i} H_i^* K H_j = \bar\lambda_i(1-(\mu/\bar\mu)^{2n})\Gamma_{ij}, \\ \sum_{k=1}^{2n} \mathcal{N}_{ik}\mathcal{M}_{k,2n+j} &= \sum_{k=1}^{2n} (\bar\lambda_j/\bar\lambda_i)^{k-1} H_i^* K \Omega^n \bar H_j \\ &= 2n\delta_{ij}\overline{H_i^T K \Omega^n H_j} = 0\end{aligned} \tag{35}$$

since $(\bar\lambda_j/\bar\lambda_i)^{2n} = 1$ and

$$\sum_{k=1}^{2n} (\bar\lambda_j/\bar\lambda_i)^{k-1} = \begin{cases} \dfrac{1-(\bar\lambda_j/\bar\lambda_i)^{2n}}{1-\bar\lambda_j/\bar\lambda_i} = 0 & i \neq j \\ 2n & i = j. \end{cases} \tag{36}$$

Likewise, we have

$$\sum_{k=1}^{2n} \mathcal{N}_{2n+i,k}\mathcal{M}_{kj} = 0, \qquad \sum_{k=1}^{2n} \mathcal{N}_{2n+i,k}\mathcal{M}_{k,2n+j} = \lambda_i(1-(\bar\mu/\mu)^{2n})\bar\Gamma_{ij}. \tag{37}$$

Note that each Γ_{ij} is an $n \times n$ matrix $(i, j = 1, \dots, 2n)$, we have

$$\det(\mathcal{N}\mathcal{M}) = |\mu|^{4n^2}(1-(\mu/\bar\mu)^{2n})^{2n^2}(1-(\bar\mu/\mu)^{2n})^{2n^2} |\det\Gamma|^2. \tag{38}$$

Since $\arg(\mu) \notin \left\{\frac{k\pi}{2n} \middle| k \in Z\right\}$, $\det\mathcal{M} \neq 0$ holds whenever $\det\Gamma \neq 0$. The lemma is proved. □

Following lemmas 3–5, we know that $\widetilde{U}(x,t,\lambda) = \lambda J + \widetilde{P}(x,t)$, $\widetilde{V}(x,t,\lambda) = \lambda^{-1}\widetilde{Q}(x,t)$ satisfies (5). Hence, (4) and remark 1 implies that $\widetilde{P} = (\widetilde{p}_i\delta_{ij})_{2n\times 2n}$, $\widetilde{Q} = (\widetilde{q}_j\delta_{i,j+1})_{2n\times 2n}$ hold where $\widetilde{p}_i$'s and $\widetilde{q}_i$'s are real functions satisfying $\widetilde{p}_{m-i} = -\widetilde{p}_i$, $\widetilde{q}_{m-1-i} = \widetilde{q}_i$ $(i = 1, 2, \dots, 2n)$.

Therefore, we get the Darboux transformation which keeps all the reductions. Now we write down the solutions more explicitly.

The solution Q is expressed in terms of G_{2n}. According to (19),

$$\begin{aligned} G_{2n} = G(x,t,0) &= \prod_{s=1}^{2n} \bar\lambda_s \left(1 - \sum_{i,j=1}^{2n} \frac{H_i \check\Gamma_{ij} H_j^* K}{\bar\lambda_j}\right) \\ &= \omega^{-n(2n-1)}\bar\mu^{2n}(1 - R^*\Gamma^{-1}S_{2n}) = -\bar\mu^{2n}(1 - R^*\Gamma^{-1}S_{2n}) \end{aligned} \tag{39}$$

where R and S_{2n} are block matrices

$$R = \begin{pmatrix} H_1^* \\ \vdots \\ H_{2n}^* \end{pmatrix}, \qquad S_{2n} = \begin{pmatrix} \bar\lambda_1^{-1} H_1^* K \\ \vdots \\ \bar\lambda_{2n}^{-1} H_{2n}^* K \end{pmatrix}. \tag{40}$$

Let

$$\xi_k = \begin{pmatrix} (\xi_k)_1 \\ \vdots \\ (\xi_k)_{2n} \end{pmatrix} = (\Omega^{k-1} \otimes I_n) \left.\begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{pmatrix}\right\} 2n \quad (k = 1, \dots, 2n), \tag{41}$$

where $(\xi_k)_i = \omega^{-(k-1)(i-1)} I_n$, then $\xi_i^* \xi_j = 2n\delta_{ij} I_n$. Moreover,

$$\Gamma\xi_k = \xi_{-k-m+3}\alpha_k, \qquad \Gamma^{-1}\xi_k = \xi_{-k-m+3}\alpha_{-k-m+3}^{-1} \tag{42}$$

where

$$\alpha_k = \sum_{j=1}^{2n} \frac{H^* K \Omega^j H}{\bar{\mu} + \omega^j \mu} \omega^{-(k-1)j} \tag{43}$$

with $\alpha_k^* = \alpha_{-k-m+3}$.

Write $H = \begin{pmatrix} h_1 \\ \vdots \\ h_{2n} \end{pmatrix}$ where $h_1, \ldots, h_{2n}$ are $1 \times n$ matrices. The ith column of R is $\xi_{2-i} h_i^*$ and the jth column of S_{2n} is $\bar{\mu}^{-1} \xi_{j-m+1} h_{m-j}^*$. Hence, we have $-\bar{\mu}^{-2n} G_{2n} = 1 - R^* \Gamma^{-1} S_{2n} = \mathrm{diag}(\eta_1, \ldots, \eta_{2n})$ where

$$\eta_j = 1 - 2n\bar{\mu}^{-1} h_j \alpha_{2-j}^{-1} h_{m-j}^*. \tag{44}$$

By (43),

$$\alpha_{2-j} = \sum_{s=1}^{2n} \frac{H^* K \Omega^s H}{\bar{\mu} + \omega^s \mu} \omega^{(j-1)s} = \sum_{k=1}^{2n} \sum_{s=1}^{2n} \frac{\omega^{(j-k)s}}{\bar{\mu} + \omega^s \mu} h_{m-k}^* h_k. \tag{45}$$

Let θ be a constant with $|\theta| < 1$. Then

$$\begin{aligned} \sum_{s=1}^{2n} \frac{\omega^{(j-k)s}}{\bar{\mu} + \theta\omega^s \mu} &= \bar{\mu}^{-1} \sum_{s=1}^{2n} \sum_{l=0}^{\infty} \omega^{(j-k+l)s} (-\theta\mu/\bar{\mu})^l \\ &= 2n\bar{\mu}^{-1} \sum_{l \geqslant 0,\, j-k+l \equiv 0 \bmod 2n} (-\theta\mu/\bar{\mu})^l. \end{aligned} \tag{46}$$

Let $l = 2n\sigma + \{k - j\}$ where $\{k\}$ denotes the remainder of k divided by $2n$, then the above equality equals to

$$2n\bar{\mu}^{-1} \sum_{\sigma \geqslant 0} (-\theta\mu/\bar{\mu})^{2n\sigma + \{k-j\}} = 2n\bar{\mu}^{-1} \frac{(-\theta\mu/\bar{\mu})^{\{k-j\}}}{1 - (-\theta\mu/\bar{\mu})^{2n}}. \tag{47}$$

Let $\theta \to 1 - 0$, then we have

$$\sum_{s=1}^{2n} \frac{\omega^{(j-k)s}}{\bar{\mu} + \omega^s \mu} = 2n\bar{\mu}^{-1} (-1)^{k-j} \frac{\mu^{\{k-j\}} \bar{\mu}^{2n-\{k-j\}}}{\bar{\mu}^{2n} - \mu^{2n}}. \tag{48}$$

Hence we get the expression of α_{2-j} as

$$\alpha_{2-j} = 2n\bar{\mu}^{-1} \sum_{k=1}^{2n} (-1)^{k-j} \frac{\mu^{\{k-j\}} \bar{\mu}^{2n-\{k-j\}}}{\bar{\mu}^{2n} - \mu^{2n}} h_{m-k}^* h_k. \tag{49}$$

Lemma 5 implies that η_j's are real. Lemma 2 gives the transformation of the solution

$$\widetilde{q}_j = \frac{\eta_{j+1}}{\eta_j} q_j. \tag{50}$$

From $q_{m-1-j} = q_j$, $\widetilde{q}_{m-1-j} = \widetilde{q}_j$, we know that $\eta_j \eta_{m-j}$ is independent of j. Considering the specific expressions of q_j, we get the transformations of the solutions $u_1, \ldots, u_l$ for both cases $C_l^{(1)}$ and $D_{l+1}^{(2)}$.

Case $C_l^{(1)}$: $n = l + 1, m = 0$. We have $\eta_j \eta_{-j} = \eta_{l+1}^2 = \eta_{2l+2}^2$ $(j = 1, 2, \ldots, 2l + 2)$. According to (10), the transformation is

$$\widetilde{u}_j = u_j + \ln \frac{\eta_j}{\eta_{2l+2}} = u_j + \frac{1}{2} \ln \frac{\eta_j}{\eta_{2l+2-j}} \quad (j = 1, \ldots, l). \tag{51}$$

Case $D_{l+1}^{(2)}$: $n = l, m = 1$. We have $\eta_j\eta_{1-j} = \eta_l\eta_{l+1} = \eta_1\eta_{2l}$ $(j = 1, 2, \ldots, 2l)$. The transformation is

$$\widetilde{u}_j = u_j + \ln\frac{\eta_j}{\sqrt{\eta_1\eta_{2l}}} = u_j + \frac{1}{2}\ln\frac{\eta_j}{\eta_{2l+1-j}} \quad (j = 1, \ldots, l). \tag{52}$$

In summary, we have the following theorems.

Theorem 1. *Suppose* $(u_1, \ldots, u_l)$ *is a solution of the two-dimensional* $C_l^{(1)}$ *Toda equation (11),* μ *is a nonzero complex number such that* $\arg(\mu) \neq \frac{k\pi}{2l+2}$ *for any integer* k*. Let* $H = \begin{pmatrix} h_1 \\ \vdots \\ h_{2l+2} \end{pmatrix}$ *be a* $(2l+2)\times(l+1)$ *solution of the Lax pair (6) with* $\lambda = \mu$ *where* p_j*'s and* q_j*'s are given by (10). Let*

$$\eta_j = 1 - h_j\left(\sum_{k=1}^{2l+2}(-1)^{k-j}\frac{\mu^{\{k-j\}}\bar{\mu}^{2l+2-\{k-j\}}}{\bar{\mu}^{2l+2} - \mu^{2l+2}}h_{2l+2-k}^*h_k\right)^{-1}h_{2l+2-j}^* \tag{53}$$

$(j = 1, 2, \ldots, 2l+2)$*. Then*

$$\widetilde{u}_j = u_j + \frac{1}{2}\ln\frac{\eta_j}{\eta_{2l+2-j}} \quad (j = 1, \ldots, l) \tag{54}$$

gives a new solution of the two-dimensional $C_l^{(1)}$ *Toda equation (11).*

Theorem 2. *Suppose* $(u_1, \ldots, u_l)$ *is a solution of the two-dimensional* $D_{l+1}^{(2)}$ *Toda equation (15),* μ *is a nonzero complex number such that* $\arg(\mu) \neq \frac{k\pi}{2l}$ *for any integer* k*. Let* $H = \begin{pmatrix} h_1 \\ \vdots \\ h_{2l} \end{pmatrix}$ *be a* $2l \times l$ *solution of the Lax pair (6) with* $\lambda = \mu$ *where* p_j*'s and* q_j*'s are given by (14). Let*

$$\eta_j = 1 - h_j\left(\sum_{k=1}^{2l}(-1)^{k-j}\frac{\mu^{\{k-j\}}\bar{\mu}^{2l-\{k-j\}}}{\bar{\mu}^{2l} - \mu^{2l}}h_{2l+1-k}^*h_k\right)^{-1}h_{2l+1-j}^* \tag{55}$$

$(j = 1, 2, \ldots, 2l)$*. Then*

$$\widetilde{u}_j = u_j + \frac{1}{2}\ln\frac{\eta_j}{\eta_{2l+1-j}} \quad (j = 1, \ldots, l) \tag{56}$$

gives a new solution of the two-dimensional $D_{l+1}^{(2)}$ *Toda equation (15).*

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